

LOG-MAJORIZATION AND LIE-TROTTER FORMULA FOR THE CARTAN BARYCENTER ON PROBABILITY MEASURE SPACES

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ABSTRACT. We extend Ando-Hiai's log-majorization for the weighted geometric mean of positive definite matrices into that for the Cartan barycenter in the general setting of probability measures on the Riemannian manifold of positive definite matrices equipped with trace metric. We also derive a version of Lie-Trotter formula and related unitarily invariant norm inequalities for the Cartan barycenter as the main application of log-majorization.

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1. INTRODUCTION

Let A be an $m \times m$ positive definite matrix with eigenvalues $\lambda_j(A)$, $1 \leq j \leq m$, arranged in decreasing order, i.e., $\lambda_1(A) \geq \cdots \geq \lambda_m(A)$ with counting multiplicities. The *log-majorization* $A \prec_{\log} B$ between positive definite matrices A and B is defined if

$$\prod_{i=1}^k \lambda_i(A) \leq \prod_{i=1}^k \lambda_i(B) \quad \text{for } 1 \leq k \leq m-1, \text{ and } \det A = \det B.$$

The log-majorization gives rise to powerful devices in deriving various norm inequalities and has many important applications such as operator means, operator monotone functions, statistical mechanics, quantum information theory and eigenvalue analysis. For instance, $A \prec_{\log} B$ implies $|||A||| \leq |||B|||$ for all unitarily invariant norms $||| \cdot |||$.

As a complementary counterpart of the Golden-Thompson trace inequality, Ando and Hiai [2] established the log-majorization on the matrix geometric mean of two

positive definite matrices: for positive definite matrices A, B and $0 \leq \alpha \leq 1$,

$$A^t \#_{\alpha} B^t \prec_{\log} (A \#_{\alpha} B)^t, \quad t \geq 1,$$

where $A \#_{\alpha} B := A^{1/2}(A^{-1/2}BA^{-1/2})^{\alpha}A^{1/2}$, the α -weighted geometric mean of A and B . This provides various norm inequalities for unitarily invariant norms via the Lie-Trotter formula $\lim_{t \rightarrow 0} (A^t \#_{\alpha} B^t)^{\frac{1}{t}} = e^{(1-\alpha)\log A + \alpha \log B}$. For instance, $|||(A^t \#_{\alpha} B^t)^{\frac{1}{t}}|||$ increases to $|||e^{(1-\alpha)\log A + \alpha \log B}|||$ as $t \searrow 0$ for any unitarily invariant norm. Ando-Hiai's log-majorization has many important applications in matrix analysis and inequalities, together with Araki's log-majorization [3] extending the Lieb-Thirring and the Golden-Thompson trace inequalities.

The matrix geometric mean $A \#_{\alpha} B$, that plays the central role in Ando-Hiai's log-majorization, appears as the unique (up to parametrization) geodesic curve $\alpha \mapsto A \#_{\alpha} B$ between A and B on the Riemannian manifold \mathbb{P}_m of positive definite matrices of size m , an important example of Cartan-Hadamard Riemannian manifolds. Alternatively, the geometric mean $A \#_{\alpha} B$ is the Cartan barycenter of the finitely supported measure $(1 - \alpha)\delta_A + \alpha\delta_B$ on \mathbb{P}_m , which is defined as the unique minimizer of the least squares problem with respect to the Riemannian distance d (see Section 2 for definition). Indeed, for a general probability measure μ on \mathbb{P}_m with finite second moment, the Cartan barycenter of μ is defined as the unique minimizer as follows:

$$G(\mu) := \arg \min_{Z \in \mathbb{P}_m} \int_{\mathbb{P}_m} d^2(Z, X) d\mu(X)$$

(see Section 2 for more details). In particular, when $\mu = \sum_{i=1}^k w_i \delta_{A_i}$ is a discrete probability measure supported on finite $A_1, \dots, A_k \in \mathbb{P}_m$, the Cartan barycenter $G(\mu)$ is the *Karcher mean* of A_1, \dots, A_k , which has extensively been discussed in these years by many authors as a multivariable extension of the geometric mean (see [6, 15, 19] and references therein).

The first aim of this paper is to establish the log-majorization (Theorem 3.6) for the Cartan barycenter in the general setting of probability measures in the Wasserstein space $\mathcal{P}^2(\mathbb{P}_m)$, the probability measures on \mathbb{P}_m with finite second moments. In this way, we generalize the log-majorization in [2] (as mentioned above) and in [9] (for the Karcher mean of multivariables) to the setting of probability measures. Our second

aim is to derive the Lie-Trotter formula (Theorem 4.8) for the Cartan barycenter

$$\lim_{t \rightarrow 0} G(\mu^t)^{\frac{1}{t}} = \exp \int_{\mathbb{P}_m} \log A \, d\mu(A)$$

under a certain integrability assumption on μ , where μ^t is the t th power of the measure μ inherited from the matrix powers on \mathbb{P}_m . Moreover, to demonstrate the usefulness of our log-majorization, we obtain several unitarily invariant norm inequalities (Corollary 4.9) based on the above Lie-Trotter formula.

The main tools of the paper involve the theory of nonpositively curved metric spaces and techniques from probability measures on metric spaces and the recent combination of the two (see [17, 1, 18]). Not only are these tools crucial for our developments, but also, we believe, significantly enhance the potential usefulness of the Cartan barycenter of probability measures in matrix analysis and inequalities. They overcome the limitation to the multivariable (finite number of matrices) setting, and provide a new bridge between two different important fields of studies of matrix analysis and probability measure theory on nonpositively curved metric spaces.

2. CARTAN BARYCENTERS

Let \mathbb{H}_m be the Euclidean space of $m \times m$ Hermitian matrices equipped with the inner product $\langle X, Y \rangle := \text{Tr}(XY)$. The *Frobenius norm* $\|\cdot\|_2$ defined by $\|X\|_2 = (\text{tr } X^2)^{1/2}$ for $X \in \mathbb{H}_m$ gives rise to the Riemannian structure on the open convex cone \mathbb{P}_m of $m \times m$ positive definite matrices with the metric

$$\langle X, Y \rangle_A := \text{Tr}(A^{-1}XA^{-1}Y), \quad A \in \mathbb{P}_m, \, X, Y \in \mathbb{H}_m,$$

where the tangent space of \mathbb{P}_m at any point $A \in \mathbb{P}_m$ is identified with \mathbb{H}_m . Then \mathbb{P}_m is a *Cartan-Hadamard Riemannian manifold*, a simply connected complete Riemannian manifold with non-positive sectional curvature (the canonical 2-tensor is non-negative). The *Riemannian trace metric* on \mathbb{P}_m is given by

$$d(A, B) := \left\| \log A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right\|_2,$$

and the unique (up to parametrization) geodesic shortest curve containing A and B is $t \in [0, 1] \mapsto A \#_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^t A^{\frac{1}{2}}$. See [14, 4] for more about these Riemannian structures.

Let $\mathcal{B} := \mathcal{B}(\mathbb{P}_m)$ be the algebra of Borel sets, the smallest σ -algebra containing the open sets of \mathbb{P}_m . We note that the Euclidean topology on \mathbb{P}_m coincides with the metric topology of the trace metric d . Let $\mathcal{P} = \mathcal{P}(\mathbb{P}_m)$ be the set of all probability measures on $(\mathbb{P}, \mathcal{B})$ and $\mathcal{P}_0 = \mathcal{P}_0(\mathbb{P}_m)$ the set of all $\mu \in \mathcal{P}$ of the form $\mu = (1/n) \sum_{j=1}^n \delta_{A_j}$, where δ_A is the point measure of mass 1 at $A \in \mathbb{P}$. For $p \in [1, \infty)$ let $\mathcal{P}^p = \mathcal{P}^p(\mathbb{P}_m)$ be the set of probability measures with *finite p -moment*, i.e., for some (and hence all) $Y \in \mathbb{P}_m$,

$$\int_{\mathbb{P}_m} d^p(X, Y) d\mu(X) < \infty.$$

We say that $\omega \in \mathcal{P}(\mathbb{P}_m \times \mathbb{P}_m)$ is a *coupling* for $\mu, \nu \in \mathcal{P}$ if μ, ν are the marginals of ω , i.e., if for all $B \in \mathcal{B}$, $\omega(B \times \mathbb{P}) = \mu(B)$ and $\omega(\mathbb{P} \times B) = \nu(B)$. We note that one such coupling is the product measure $\mu \times \nu$. We denote the set of all couplings for $\mu, \nu \in \mathcal{P}(\mathbb{P}_m)$ by $\Pi(\mu, \nu)$.

The *Wasserstein distance* d_p^W on \mathcal{P}^p is defined by

$$d_p^W(\mu, \nu) := \left[\inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{P}_m \times \mathbb{P}_m} d^p(X, Y) d\pi(X, Y) \right]^{\frac{1}{p}}.$$

It is known that d_p^W is a complete metric on \mathcal{P}^p and \mathcal{P}_0 is dense in \mathcal{P}^p [17]. Note that $\mathcal{P}_0 \subset \mathcal{P}^q \subset \mathcal{P}^p \subset \mathcal{P}^1$ and $d_p^W \leq d_q^W$ for $1 \leq p \leq q < \infty$. We note that these basic results on probability measure spaces hold in general setting of complete metric spaces in which cases separability assumption is necessary.

The following result on Lipschitz property of push-forward maps between metric spaces appears in [16], where X, Y are metric spaces and the distance d_p^W on $\mathcal{P}^p(X), \mathcal{P}^p(Y)$ are defined as above.

Lemma 2.1. *Let $f : X \rightarrow Y$ be a Lipschitz map with Lipschitz constant C . Then the push-forward map $f_* : \mathcal{P}^p(X) \rightarrow \mathcal{P}^p(Y)$, $f_*(\mu) = \mu \circ f^{-1}$, is Lipschitz with respect to d_p^W with Lipschitz constant C for $1 \leq p < \infty$.*

Definition 2.2. The *Cartan barycenter* map $G : \mathcal{P}^1(\mathbb{P}_m) \rightarrow \mathbb{P}_m$ is defined by

$$G(\mu) = \arg \min_{Z \in \mathbb{P}} \int_{\mathbb{P}} [d^2(Z, X) - d^2(Y, X)] d\mu(X), \quad \mu \in \mathcal{P}^1(\mathbb{P}_m),$$

for a fixed Y . The uniqueness and existence of the minimizer is well-known and the unique minimizer is independent of Y (see [17, Proposition 4.3]). On $\mathcal{P}^2(\mathbb{P}_m)$, the

Cartan barycenter is determined by

$$G(\mu) = \arg \min_{Z \in \mathbb{P}} \int_{\mathbb{P}} d^2(Z, X) d\mu(X).$$

The following contraction property appears in [17].

Theorem 2.3 (Fundamental Contraction Property). *For every $\mu, \nu \in \mathcal{P}^p$,*

$$d(G(\mu), G(\nu)) \leq d_p^W(\mu, \nu).$$

For $\mu = (1/n) \sum_{j=1}^n \delta_{A_j} \in \mathcal{P}_0(\mathbb{P}_m)$, we denote

$$G(A_1, \dots, A_n) := G(\mu).$$

3. LOG-MAJORIZATION

For $1 \leq k \leq m$ and $A \in \mathbb{P}_m$, let $\Lambda^k A$ be the k th *antisymmetric tensor power* of A . See [2, 7, 10] for basic properties of Λ^k ; for instance

$$\Lambda^k(AB) = (\Lambda^k A)(\Lambda^k B), \tag{3.1}$$

$$\Lambda^k(A^t) = (\Lambda^k A)^t, \quad t > 0, \tag{3.2}$$

$$\lambda_1(\Lambda^k A) = \prod_{j=1}^k \lambda_j(A). \tag{3.3}$$

The k th antisymmetric tensor power map Λ^k maps \mathbb{P}_m continuously into \mathbb{P}_ℓ where $\ell := \binom{m}{k}$. This induces the push-forward map

$$\Lambda_*^k : \mathcal{P}^p(\mathbb{P}_m) \rightarrow \mathcal{P}^p(\mathbb{P}_\ell), \quad \Lambda_*^k(\mu) := \mu \circ (\Lambda^k)^{-1},$$

that is, $\Lambda_*^k(\mu)(\mathcal{O}) = \mu((\Lambda^k)^{-1}(\mathcal{O}))$ for all Borel sets \mathcal{O} .

The following shows, in particular, the continuity of the push-forward map of the antisymmetric tensor power map.

Proposition 3.1. *The map $\Lambda^k : \mathbb{P}_m \rightarrow \mathbb{P}_\ell$ is Lipschitzian, that is,*

$$d(\Lambda^k A, \Lambda^k B) \leq \alpha_{m,k} d(A, B), \quad A, B \in \mathbb{P}_m,$$

where $\alpha_{m,k} := \sqrt{k \binom{m-1}{k-1}}$. Furthermore, $\Lambda_*^k : \mathcal{P}^p(\mathbb{P}_m) \rightarrow \mathcal{P}^p(\mathbb{P}_\ell)$ is Lipschitzian for every $p \geq 1$, that is,

$$d_p^W(\Lambda_*^k(\mu), \Lambda_*^k(\nu)) \leq \alpha_{m,k} d_p^W(\mu, \nu), \quad \mu, \nu \in \mathcal{P}^p(\mathbb{P}_m).$$

Proof. The eigenvalue list of $\Lambda^k(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) = (\Lambda^k A)^{-\frac{1}{2}}(\Lambda^k B)(\Lambda^k A)^{-\frac{1}{2}}$ is

$$\prod_{j=1}^k \lambda_{i_j}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}), \quad 1 \leq i_1 < \dots < i_k \leq m.$$

Hence

$$\begin{aligned} d^2(\Lambda^k A, \Lambda^k B) &= \|\log \Lambda^k(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\|_2^2 \\ &= \sum_{1 \leq i_1 < \dots < i_k \leq m} \log^2 \left(\prod_{j=1}^k \lambda_{i_j}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \right) \\ &= \sum_{1 \leq i_1 < \dots < i_k \leq m} \left[\sum_{j=1}^k \log \lambda_{i_j}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \right]^2 \\ &\leq \sum_{1 \leq i_1 < \dots < i_k \leq m} k \sum_{j=1}^k \log^2 \lambda_{i_j}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \\ &= k \binom{m-1}{k-1} \sum_{i=1}^m \log^2 \lambda_i(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \\ &= k \binom{m-1}{k-1} d^2(A, B). \end{aligned}$$

The Lipschitz continuity of Λ_*^k follows by Lemma 2.1. □

Proposition 3.2. *For $p \geq 1$, the following diagram commute*

$$\begin{array}{ccc} \mathbb{P}_m & \xrightarrow{\Lambda^k} & \mathbb{P}_\ell \\ G \uparrow & & \uparrow G \\ \mathcal{P}^p(\mathbb{P}_m) & \xrightarrow{\Lambda_*^k} & \mathcal{P}^p(\mathbb{P}_\ell) \end{array},$$

that is,

$$G \circ \Lambda_*^k = \Lambda^k \circ G. \tag{3.4}$$

Proof. We first consider the case $\mu = (1/n) \sum_{j=1}^n \delta_{A_j} \in \mathcal{P}_0(\mathbb{P}_m)$. One can directly see that

$$\Lambda_*^k(\mu) = \frac{1}{n} \sum_{j=1}^n \delta_{\Lambda^k A_j}.$$

This, together with the result of Bhatia and Karandikar [7, Theorem 4.4], implies that

$$\begin{aligned} G(\Lambda_*^k(\mu)) &= G(\Lambda^k A_1, \dots, \Lambda^k A_n) \\ &= \Lambda^k G(A_1, \dots, A_n) = \Lambda^k G(\mu). \end{aligned}$$

This shows that our claim holds on $\mathcal{P}_0(\mathbb{P}_m)$.

By continuity of G , Λ_*^k and Λ^k (Theorem 2.3 and Proposition 3.1) and density of $\mathcal{P}^0(\mathbb{P}_m)$ in $\mathcal{P}^2(\mathbb{P}_m)$, the result follows. \square

Next, we introduce powers of probability measures on \mathbb{P}_m .

Definition 3.3. For $t \in \mathbb{R} \setminus \{0\}$ and $\mathcal{O} \in \mathcal{B}(\mathbb{P}_m)$, we let $\mathcal{O}^t := \{A^t : A \in \mathcal{O}\}$ and

$$\mu^t(\mathcal{O}) := \mu(\mathcal{O}^{\frac{1}{t}}).$$

Note that $\mu^t \in \mathcal{P}^p(\mathbb{P}_m)$ if $\mu \in \mathcal{P}^p(\mathbb{P}_m)$. In terms of push-forward measures,

$$\mu^t = g_* \mu,$$

where $g(X) := X^t$. By (3.2) and the definition of push-forward map, we have

$$\Lambda_*^k(\mu^t) = \Lambda_*^k(\mu)^t \quad \text{for } \mu \in \mathcal{P}^p(\mathbb{P}_m), \quad t \neq 0. \quad (3.5)$$

We write $\|\cdot\|$ for the operator norm. The following inequality appears in [13, Corollary 4.4].

Theorem 3.4. For every $\mu \in \mathcal{P}^2(\mathbb{P}_m)$ and $t \geq 1$,

$$\|G(\mu^t)\| \leq \|G(\mu)^t\|. \quad (3.6)$$

Remark 3.5. Inequality (3.6) in the above follows from the fact that for any $t \geq 1$,

$$G(\mu) \leq I \implies G(\mu^t) \leq I,$$

which is one of important consequences of the extension of Yamazaki's inequality [19, Theorem 3] to probability measures in $\mathcal{P}^2(\mathbb{P}_m)$:

$$\int_{\mathbb{P}_m} \log X \, d\mu(X) \leq 0 \implies G(\mu) \leq I. \quad (3.7)$$

The proof of (3.7) depends heavily on the monotonicity of the Cartan barycenter [15, 13] and its relation to the Karcher equation on $\mathcal{P}^2(\mathbb{P}_m)$:

$$Z = G(\mu) \iff \int_{\mathbb{P}_m} \log(Z^{-\frac{1}{2}} X Z^{-\frac{1}{2}}) d\mu(X) = 0.$$

The main result of this section is the following:

Theorem 3.6. *For every $\mu \in \mathcal{P}^2(\mathbb{P}_m)$ and $t \geq 1$,*

$$G(\mu^t) \prec_{\log} G(\mu)^t.$$

In particular, for any unitary invariant norm $||| \cdot |||$,

$$|||G(\mu^t)||| \leq |||G(\mu)^t|||, \quad t \geq 1.$$

Proof. For $1 \leq k \leq m$ we have

$$\begin{aligned} \prod_{j=1}^k \lambda_j(G(\mu^t)) &= \lambda_1(\Lambda^k G(\mu^t)) = \|\Lambda^k G(\mu^t)\| \\ &= \|G(\Lambda_*^k(\mu^t))\| = \|G((\Lambda_*^k(\mu))^t)\| \\ &\leq \|G((\Lambda_*^k(\mu)))\|^t = \prod_{j=1}^k \lambda_j(G(\mu)^t), \end{aligned}$$

where (3.3), (3.4), (3.5) and (3.6) have been used. It remains to show that $\det G(\mu^t) = \det G(\mu)^t$. It remains to show that $\det G(\mu^t) = \det G(\mu)^t$. When $k = m$, since $\Lambda^m = \det$ and $G((\Lambda_*^m(\mu))^t)$ is a positive scalar, the equalities shown above say that $\det G(\mu^t) = G((\Lambda_*^m(\mu))^t)$. In the one-dimensional case on $\mathbb{P}_1 = (0, \infty)$, we find by a direct computation that

$$G(\nu) = \exp \int_{(0, \infty)} \log x \, d\nu(x)$$

for every $\nu \in \mathcal{P}^2((0, \infty))$. Therefore,

$$G((\Lambda_*^m(\mu))^t) = \exp \int_{(0, \infty)} \log x \, d(\Lambda_*^m(\mu)^t)(x) = \exp \int_{\mathbb{P}_m} \log(\det^t A) \, d\mu(A)$$

$$\begin{aligned}
&= \exp \int_{\mathbb{P}_m} t \operatorname{tr}(\log A) d\mu(A) = \det^t \left(\exp \int_{\mathbb{P}_m} \log A d\mu(A) \right) \\
&= \det G(\mu)^t,
\end{aligned}$$

implying that $\det G(\mu^t) = \det G(\mu)^t$. \square

By a consequence of the preceding theorem, we have the following:

Corollary 3.7. *For every $\mu \in \mathcal{P}^2(\mathbb{P}_m)$,*

$$G(\mu^q)^{\frac{1}{q}} \prec_{(\log)} G(\mu^p)^{\frac{1}{p}}, \quad 0 < p \leq q,$$

$$G(\mu^p)^{\frac{1}{p}} \prec_{(\log)} G(\mu) \prec_{(\log)} G(\mu^{\frac{1}{p}})^p, \quad p \geq 1,$$

and therefore

$$|||G(\mu^q)^{\frac{1}{q}}||| \leq |||G(\mu^p)^{\frac{1}{p}}|||, \quad 0 < p \leq q, \quad (3.8)$$

$$|||G(\mu^p)^{\frac{1}{p}}||| \leq |||G(\mu)||| \leq |||G(\mu^{\frac{1}{p}})^p|||, \quad p \geq 1$$

for all unitarily invariant norms $||| \cdot |||$.

4. LIE-TROTTER FORMULA

The Lie-Trotter formula for the Cartan mean of multivariable positive definite matrices is

$$\lim_{t \rightarrow 0} G(A_1^t, \dots, A_m^t)^{\frac{1}{t}} = \exp \left(\frac{1}{n} \sum_{j=1}^n \log A_j \right).$$

See [9, 8, 5].

In this section we establish the Lie-Trotter formula and associated norm inequalities for probability measures in a certain sub-class of $\mathcal{P}^2(\mathbb{P}_m)$.

Lemma 4.1. *For every $A \in \mathbb{P}_m$,*

$$\|\log A\| \leq \log(\|A\| + \|A^{-1}\|).$$

Proof. Since $\lambda_m(A)I \leq A \leq \lambda_1(A)I$, we have

$$(\log \lambda_m(A))I \leq \log A \leq (\log \lambda_1(A))I,$$

which implies that $\|\log A\| = \max\{|\log \lambda_1(A)|, |\log \lambda_m(A)|\}$. Since

$$|\log a| = \begin{cases} \log a & \text{if } a \geq 1, \\ \log a^{-1} & \text{if } 0 < a \leq 1, \end{cases}$$

we have

$$\begin{aligned} \|\log A\| &= \max\{\log \lambda_1(A), \log \lambda_m(A)^{-1}\} \\ &= \max\{\log \|A\|, \log \|A^{-1}\|\} \leq \log(\|A\| + \|A^{-1}\|). \end{aligned}$$

□

Lemma 4.2. *For every $\theta > 0$ there exists a constant $k_\theta > 0$ such that*

$$\|\log A\|_2 \leq k_\theta (\|A\| + \|A^{-1}\|)^\theta, \quad A \in \mathbb{P}_m.$$

Proof. First, note that $\|A\| + \|A^{-1}\| \geq 2\sqrt{\|A\| \|A^{-1}\|} \geq 2$. Since

$$\lim_{x \rightarrow \infty} \frac{\log x}{x^\theta} = 0,$$

we have

$$c_\theta := \sup_{x \geq 1} \frac{\log x}{x^\theta} < \infty.$$

It follows from Lemma 4.1 that

$$\begin{aligned} \|\log A\|_2 &\leq \sqrt{m} \|\log A\| \leq \sqrt{m} \log(\|A\| + \|A^{-1}\|) \\ &\leq \sqrt{m} c_\theta (\|A\| + \|A^{-1}\|)^\theta \end{aligned}$$

for all $A \in \mathbb{P}_m$.

□

Now, for $\mu \in \mathcal{P}(\mathbb{P}_m)$ we consider the condition

$$\int_{\mathbb{P}_m} (\|A\| + \|A^{-1}\|) d\mu(A) < \infty. \quad (4.1)$$

Lemma 4.3. *If $\mu \in \mathcal{P}(\mathbb{P}_m)$ satisfies (4.1), then $\mu \in \mathcal{P}^p(\mathbb{P}_m)$ for every $p \in [1, \infty)$.*

Proof. By Lemma 4.2 with $\theta = 1/p$ we have

$$\|\log A\|_2 \leq k_{1/p} (\|A\| + \|A^{-1}\|)^{1/p}, \quad A \in \mathbb{P}_m.$$

Therefore,

$$\int_{\mathbb{P}_m} d^p(A, I) d\mu(A) = \int_{\mathbb{P}_m} \|\log A\|_2^p d\mu(A) \leq k_{1/p}^p \int_{\mathbb{P}_m} (\|A\| + \|A^{-1}\|) d\mu(A) < \infty,$$

implying $\mu \in \mathcal{P}^p(\mathbb{P}_m)$. \square

By assumption (4.1) one can define the arithmetic and harmonic means of μ as

$$\int_{\mathbb{P}_m} A d\mu(A), \quad \left(\int_{\mathbb{P}_m} A^{-1} d\mu(A) \right)^{-1},$$

respectively. By Lemma 4.2 one can also define the Cartan barycenter $G(\mu)$.

The next lemma will be crucial in the proof of our main result of this section.

Lemma 4.4. *Assume that $\mu \in \mathcal{P}(\mathbb{P}_m)$ satisfies (4.1). Then there exist a sequence $\{\mu_n\}_{n=1}^\infty$ in $\mathcal{P}_0(\mathbb{P}_m)$ such that, as $n \rightarrow \infty$,*

$$d_p^W(\mu_n, \mu) \longrightarrow 0 \quad \text{for all } p \in [1, \infty),$$

and

$$\int_{\mathbb{P}_m} A d\mu_n(A) \longrightarrow \int_{\mathbb{P}_m} A d\mu(A), \quad \int_{\mathbb{P}_m} A^{-1} d\mu_n(A) \longrightarrow \int_{\mathbb{P}_m} A^{-1} d\mu(A).$$

Proof. Let $\mu \in \mathcal{P}(\mathbb{P}_m)$ satisfies (4.1). By Lemma 4.3, $\mu \in \mathcal{P}^p(\mathbb{P}_m)$ for all $p \in [1, \infty)$. From a basic fact on the convergence in Wasserstein spaces (see [1, Proposition 7.1.5], [18, Theorem 7.12 and (5.1.20)]), we see that for a sequence μ_n in $\mathcal{P}_0(\mathbb{P}_m)$, $\lim_{n \rightarrow \infty} d_p^W(\mu_n, \mu) \rightarrow 0$ holds for all $p \in [1, \infty)$ if and only if μ_n converges to μ weakly and

$$\sup_n \int_{\mathbb{P}_m} d^p(A, A_0) d\mu_n(A) < \infty \tag{4.2}$$

for all $p \in [1, \infty)$ and for some (equivalently, for any) $A_0 \in \mathbb{P}_m$. Here, note that if (4.2) holds, then for any $p' \in (0, p)$ we have

$$\sup_n \int_{\{d(A, A_0) \geq R\}} d^{p'}(A, A_0) d\mu_n(A) \leq \frac{1}{R^{p-p'}} \sup_n \int_{\mathbb{P}_m} d^p(A, A_0) d\mu_n(A) \longrightarrow 0$$

as $R \rightarrow \infty$.

Now, we construct a desired sequence in $\mathcal{P}_0(\mathbb{P}_m)$ converging to μ . For each $n \in \mathbb{N}$ let

$$\Gamma_n := \{A \in \mathbb{P}_m : n^{-1}I \leq A \leq nI\}$$

and $\alpha_n := \mu(\mathbb{P}_m \setminus \Gamma_n)$; hence $\alpha_n \rightarrow 0$. For any $B \in \Gamma_n$ define

$$\mathcal{O}_{n,B} := \{A \in \mathbb{P}_m : \|A - B\| < (n+1)^{-1}\|B\|, \|A^{-1} - B^{-1}\| < (n+1)^{-1}\|B^{-1}\|\},$$

which is an open subset of \mathbb{P}_m with respect to the operator norm distance (also with respect to d). Since Γ_n is compact, there exists a finite set $\{B_{n,j}\}_{j=1}^{k_n}$ in Γ_n such that $\Gamma_n \subset \bigcup_{j=1}^{k_n} \mathcal{O}_{B_{n,j}}$. We then have a Borel partition $\{\mathcal{Q}_{n,j}\}_{j=1}^{k_n}$ of Γ_n such that $\mathcal{Q}_{n,j} \subset \mathcal{O}_{B_{n,j}}$ for $1 \leq j \leq k_n$. Here, we can iteratively construct $\{\mathcal{Q}_{n,j}\}_{j=1}^{k_n}$, $n \in \mathbb{N}$, in the following way: $\{\mathcal{Q}_{n+1,j}\}_{j=1}^{k_{n+1}}$ refines $\{\mathcal{Q}_{n,j}\}_{j=1}^{k_n}$, that is, each $\mathcal{Q}_{n,j}$ is a union of some $\mathcal{Q}_{n+1,l}$'s. Now, define $\mu_n \in \mathcal{P}_0(\mathbb{P}_m)$ as

$$\mu_n := \sum_{j=1}^{k_n} \mu(\mathcal{Q}_{n,j}) \delta_{B_{n,j}} + \alpha_n \delta_I.$$

Note that when $A \in \mathcal{Q}_{n,j}$ we have

$$\|A\| \geq \|B_{n,j}\| - \|A - B_{n,j}\| \geq \left(1 - \frac{1}{n+1}\right) \|B_{n,j}\| \geq \frac{1}{2} \|B_{n,j}\|, \quad (4.3)$$

$$\|A^{-1}\| \geq \|B_{n,j}^{-1}\| - \|A^{-1} - B_{n,j}^{-1}\| \geq \left(1 - \frac{1}{n+1}\right) \|B_{n,j}^{-1}\| \geq \frac{1}{2} \|B_{n,j}^{-1}\|, \quad (4.4)$$

so that

$$\|A - B_{n,j}\| \leq \frac{1}{n+1} \|B_{n,j}\| \leq \frac{2}{n+1} \|A\|, \quad (4.5)$$

$$\|A^{-1} - B_{n,j}^{-1}\| \leq \frac{1}{n+1} \|B_{n,j}^{-1}\| \leq \frac{2}{n+1} \|A^{-1}\|. \quad (4.6)$$

By (4.5) we estimate

$$\begin{aligned} & \left\| \int_{\mathbb{P}_m} A d\mu(A) - \int_{\mathbb{P}_m} A d\mu_n(A) \right\| \\ & \leq \left\| \int_{\Gamma_n} A d\mu(A) - \int_{\Gamma_n} A d\mu_n(A) \right\| + \left\| \int_{\mathbb{P}_m \setminus \Gamma_n} A d\mu(A) \right\| \\ & \leq \sum_{j=1}^{k_n} \int_{\mathcal{Q}_{n,j}} \|A - B_{n,j}\| d\mu(A) + \alpha_n + \int_{\mathbb{P}_m \setminus \Gamma_n} \|A\| d\mu(A) \\ & \leq \frac{2}{n+1} \sum_{j=1}^{k_n} \int_{\mathcal{Q}_{n,j}} \|A\| d\mu(A) + \alpha_n + \int_{\mathbb{P}_m \setminus \Gamma_n} \|A\| d\mu(A) \\ & \leq \frac{2}{n+1} \int_{\mathbb{P}_m} \|A\| d\mu(A) + \alpha_n + \int_{\mathbb{P}_m \setminus \Gamma_n} \|A\| d\mu(A) \\ & \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

By (4.6) we similarly estimate

$$\begin{aligned}
& \left\| \int_{\mathbb{P}_m} A^{-1} d\mu(A) - \int_{\mathbb{P}_m} A^{-1} d\mu_n(A) \right\| \\
& \leq \frac{2}{n+1} \int_{\mathbb{P}_m} \|A^{-1}\| d\mu(A) + \alpha_n + \int_{\mathbb{P}_m \setminus \Gamma_n} \|A^{-1}\| d\mu(A) \\
& \longrightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

For each $p \in [1, \infty)$, by Lemma 4.2 with $\theta = 1/p$ we have

$$\begin{aligned}
\int_{\mathbb{P}_m} \|\log A\|_2^p d\mu_n(A) & \leq k_{1/p}^p \int_{\mathbb{P}_m} (\|A\| + \|A^{-1}\|) d\mu_n(A) \\
& = k_{1/p}^p \left[\sum_{j=1}^{k_n} (\|B_{n,j}\| + \|B_{n,j}^{-1}\|) \mu(\mathcal{Q}_{n,j}) + 2\alpha_n \right] \\
& \leq k_{1/p}^p \left[\sum_{j=1}^{k_n} \int_{\mathcal{Q}_{n,j}} 2(\|A\| + \|A^{-1}\|) d\mu(A) + 2 \right] \\
& = 2k_{1/p}^p \left[\int_{\mathbb{P}_m} (\|A\| + \|A^{-1}\|) d\mu(A) + 1 \right],
\end{aligned}$$

where we have used (4.3) and (4.4) for the second inequality. Therefore,

$$\sup_n \int_{\mathbb{P}_m} d^p(A, I) d\mu_n(A) < \infty. \tag{4.7}$$

Furthermore, let f be a bounded continuous real function on \mathbb{P}_m , and let $n \geq n_0$. We estimate

$$\begin{aligned}
& \left| \int_{\mathbb{P}_m} f(A) d\mu(A) - \int_{\mathbb{P}_m} f(A) d\mu_n(A) \right| \\
& \leq \left| \int_{\Gamma_{n_0}} f(A) d\mu(A) - \int_{\Gamma_{n_0}} f(A) d\mu_n(A) \right| + \|f\| \{ \mu(\mathbb{P}_m \setminus \Gamma_{n_0}) + \mu_n(\mathbb{P}_m \setminus \Gamma_{n_0}) \} \\
& \leq \sum_{\mathcal{Q}_{n,j} \subset \Gamma_{n_0}} \int_{\mathcal{Q}_{n,j}} |f(A) - f(B_{n,j})| d\mu(A) + 2\|f\|\alpha_{n_0} \\
& \leq \sup \left\{ |f(A) - f(A')| : A, A' \in \Gamma_{n_0}, \|A - A'\| \leq \frac{n_0}{n+1} \right\} + 2\|f\|\alpha_{n_0},
\end{aligned}$$

where we have used (4.5) for the last inequality. Since f is uniformly continuous on Γ_{n_0} , the above estimate implies that

$$\limsup_{n \rightarrow \infty} \left| \int_{\mathbb{P}_m} f(A) d\mu(A) - \int_{\mathbb{P}_m} f(A) d\mu_n(A) \right| \leq 2\|f\|_{\alpha_{n_0}}.$$

Letting $n_0 \rightarrow \infty$ gives

$$\lim_{n \rightarrow \infty} \int_{\mathbb{P}_m} f(A) d\mu_n(A) = \int_{\mathbb{P}_m} f(A) d\mu(A)$$

so that μ_n converges to μ weakly. Thanks to the fact stated at the beginning of the proof, this together with (4.7) implies that $d_p^W(\mu_n, \mu) \rightarrow 0$. \square

The following *AGH* (arithmetic-geometric-harmonic) *mean inequalities* were shown for $\mu \in \mathcal{P}_0(\mathbb{P}_m)$ in [20, Theorem 2] and extended to the case of a compactly supported μ in [12]. We further extend it to the case of μ satisfying (4.1).

Proposition 4.5 (AGH inequalities). *If $\mu \in \mathcal{P}(\mathbb{P}_m)$ satisfies (4.1), then*

$$\left(\int_{\mathbb{P}_m} A^{-1} d\mu(A) \right)^{-1} \leq G(\mu) \leq \int_{\mathbb{P}_m} A d\mu(A). \quad (4.8)$$

Proof. By Lemma 4.4 choose a sequence $\{\mu_n\}$ in $\mathcal{P}_0(\mathbb{P}_m)$ such that $d_2^W(\mu_n, \mu) \rightarrow 0$ (hence $G(\mu_n) \rightarrow G(\mu)$ by Theorem 2.3) and

$$\int_{\mathbb{P}_m} A d\mu_n(A) \longrightarrow \int_{\mathbb{P}_m} A d\mu(A), \quad \int_{\mathbb{P}_m} A^{-1} d\mu_n(A) \longrightarrow \int_{\mathbb{P}_m} A^{-1} d\mu(A).$$

Since inequalities (4.8) hold for μ_n , the result follows by taking the limit of (4.8) for μ_n . \square

Remark 4.6. Let A_{ij} and $(A^{-1})_{ij}$ denote the (i, j) -entries of A, A^{-1} , respectively. Then it is clear that the functions $A \in \mathbb{P}_m \mapsto A_{ij}, (A^{-1})_{ij}$ are integrable with respect to μ for all $i, j = 1, \dots, m$ if and only if condition (4.1) holds. Hence (4.1) is the best possible assumption for the AGH mean inequalities in Proposition 4.5 to make sense.

Lemma 4.7. *For every $\mu \in \mathcal{P}(\mathbb{P}_m)$ with (4.1),*

$$\frac{1}{t} \log \int_{\mathbb{P}_m} A^t d\mu(A) \longrightarrow \int_{\mathbb{P}_m} \log A d\mu(A),$$

or equivalently,

$$\left(\int_{\mathbb{P}_m} A^t d\mu(A) \right)^{\frac{1}{t}} = \exp \int_{\mathbb{P}_m} \log A d\mu(A)$$

as $t \rightarrow 0$ with $|t| \leq 1$.

Proof. First, note that $\int_{\mathbb{P}_m} \log A \, d\mu(A)$ exists by Lemma 4.2. For any $A \in \mathbb{P}_m$ we write

$$A^t = e^{t \log A} = I + t \log A + R(t, A),$$

where

$$R(t; A) := \sum_{n=2}^{\infty} \frac{t^n}{n!} (\log A)^n.$$

Assuming $|t| \leq 1$ we have

$$\|R(t, A)\| \leq t^2 \sum_{n=0}^{\infty} \frac{1}{n!} \|\log A\|^n = t^2 e^{\|\log A\|} \leq t^2 (\|A\| + \|A^{-1}\|)$$

by Lemma 4.1. Therefore,

$$\int_{\mathbb{P}_m} \|R(t, A)\| \, d\mu(A) \leq t^2 \int_{\mathbb{P}_m} (\|A\| + \|A^{-1}\|) \, d\mu(A) = O(t^2) \quad \text{as } t \rightarrow 0,$$

so that we have

$$\int_{\mathbb{P}_m} A^t \, d\mu(A) = I + t \int_{\mathbb{P}_m} \log A \, d\mu(A) + O(t^2).$$

This implies that

$$\frac{1}{t} \log \int_{\mathbb{P}_m} A^t \, d\mu(A) = \int_{\mathbb{P}_m} \log A \, d\mu(A) + O(t),$$

and hence

$$\lim_{t \rightarrow 0} \frac{1}{t} \log \int_{\mathbb{P}_m} A^t \, d\mu(A) = \int_{\mathbb{P}_m} \log A \, d\mu(A).$$

□

Finally, for $\mu \in \mathcal{P}(\mathbb{P}_m)$ we consider the condition

$$\int_{\mathbb{P}_m} (\|A\| + \|A^{-1}\|)^r \, d\mu(A) < \infty \tag{4.9}$$

for some $r > 0$. It is obvious that if (4.9) holds for $r > 0$, then it also holds for any $r' \in (0, r]$. Moreover, condition (4.9) is equivalent to

$$\int_{\mathbb{P}_m} (\|A^r\| + \|A^{-r}\|) \, d\mu(A) < \infty,$$

i.e., μ^r satisfies (4.1).

Our main result of this section is the following:

Theorem 4.8 (Lie-Trotter formula). *Let $\mu \in \mathcal{P}(\mathbb{P}_m)$ satisfying (4.9) for some $r > 0$. Then we have*

$$\lim_{t \rightarrow 0} G(\mu^t)^{\frac{1}{t}} = \exp \int_{\mathbb{P}_m} \log A d\mu(A). \quad (4.10)$$

Proof. First, assume that μ satisfies (4.1). For any $t \neq 0$, by using Proposition 4.5 to μ^t we have

$$\begin{aligned} \left(\int_{\mathbb{P}_m} A^{-t} d\mu(A) \right)^{-1} &= \left(\int_{\mathbb{P}_m} A^{-1} d\mu^t(A) \right)^{-1} \\ &\leq G(\mu^t) \leq \int_{\mathbb{P}_m} A d\mu^t(A) = \int_{\mathbb{P}_m} A^t d\mu(A). \end{aligned}$$

Since $\log x$ is operator monotone on $(0, \infty)$, the above inequality gives

$$\begin{aligned} -\frac{1}{t} \log \int_{\mathbb{P}_m} A^{-t} d\mu(A) &\leq \log G(\mu^t)^{\frac{1}{t}} \leq \frac{1}{t} \log \int_{\mathbb{P}_m} A^t d\mu(A) \quad \text{if } t > 0, \\ -\frac{1}{t} \log \int_{\mathbb{P}_m} A^{-t} d\mu(A) &\geq \log G(\mu^t)^{\frac{1}{t}} \geq \frac{1}{t} \log \int_{\mathbb{P}_m} A^t d\mu(A) \quad \text{if } t < 0. \end{aligned}$$

From Lemma 4.7 this implies that

$$\lim_{t \rightarrow 0} \log G(\mu^t)^{\frac{1}{t}} = \int_{\mathbb{P}_m} \log A d\mu(A). \quad (4.11)$$

Next, assume that μ satisfies (4.9) for some $r > 0$, that is, μ^r satisfies (4.1). The above case yields

$$\lim_{t \rightarrow 0} \log G((\mu^r)^t)^{\frac{1}{t}} = \int_{\mathbb{P}_m} \log A d\mu^r(A).$$

Note that the left-hand side in the above is

$$\lim_{t \rightarrow 0} \log G(\mu^{rt})^{\frac{1}{t}} = r \lim_{t \rightarrow 0} \log G(\mu^t)^{\frac{1}{t}},$$

while the right-hand side is

$$r \int_{\mathbb{P}_m} \log A d\mu(A).$$

Hence we have (4.11) again, which implies (4.10). \square

The next corollary extends [5, Corollary 2] to the case of probability measures satisfying (4.9).

Corollary 4.9. *Assume that $\mu \in \mathcal{P}(\mathbb{P}_m)$ satisfies (4.9) for an $r > 0$ and $||| \cdot |||$ is any unitarily invariant norm. Then*

(a) For every $t > 0$,

$$|||G(\mu^{-t})^{-\frac{1}{t}}||| = |||G(\mu^t)^{\frac{1}{t}}||| \leq |||\exp \int_{\mathbb{P}_m} \log A d\mu(A)|||, \quad (4.12)$$

and $|||G(\mu^t)^{\frac{1}{t}}|||$ increases to $|||\exp \int_{\mathbb{P}_m} \log A d\mu(A)|||$ as $t \searrow 0$.

(b) If $0 < t \leq r$, then

$$\begin{aligned} & |||\left(\int_{\mathbb{P}_m} A^{-t} d\mu(A)\right)^{-\frac{1}{t}}||| \leq |||G(\mu^t)^{\frac{1}{t}}||| \\ & \leq |||\exp \int_{\mathbb{P}_m} \log A d\mu(A)||| \leq |||\left(\int_{\mathbb{P}_m} A^t d\mu(A)\right)^{\frac{1}{t}}|||. \end{aligned} \quad (4.13)$$

Furthermore, $|||(\int_{\mathbb{P}_m} A^t d\mu(A))^{\frac{1}{t}}|||$ decreases to $|||\exp \int_{\mathbb{P}_m} \log A d\mu(A)|||$ and $|||(\int_{\mathbb{P}_m} A^{-t} d\mu(A))^{-\frac{1}{t}}|||$ increases to $|||\exp \int_{\mathbb{P}_m} \log A d\mu(A)|||$ as $r \geq t \searrow 0$.

Proof. When $\mu \in \mathcal{P}^2(\mathbb{P}_m)$ (without condition (4.9)), from the invariance $G(\mu^{-1}) = G(\mu)^{-1}$ we find that $G(\mu^{-t})^{-\frac{1}{t}} = G(\mu^t)^{\frac{1}{t}}$, implying the equality in (4.12). It follows from (3.8) that $|||G(\mu^t)^{\frac{1}{t}}|||$ is increasing as $t \searrow 0$. In the rest, assume (4.9) for an $r > 0$.

(a) The inequality in (4.12) is immediately seen from Theorem 4.8 together with the fact shown above.

(b) Assume that $0 < t' < t \leq r$ and prove that

$$\int_{\mathbb{P}_m} A^{t'} d\mu(A) \leq \left(\int_{\mathbb{P}_m} A^t d\mu(A)\right)^{\frac{t'}{t}}, \quad (4.14)$$

$$\left(\int_{\mathbb{P}_m} A^{-t'} d\mu(A)\right)^{-1} \geq \left(\int_{\mathbb{P}_m} A^{-t} d\mu(A)\right)^{-\frac{t'}{t}}. \quad (4.15)$$

For each $n \in \mathbb{N}$ let Γ_n and α_n be as in the proof of Lemma 4.4. Since A^t and $A^{t'}$ are uniformly continuous on the compact set Γ_n , one can choose a sequence of simple functions $\sum_{j=1}^{k_\ell} B_{\ell,j} 1_{\mathcal{Q}_{\ell,j}}$, $\ell \in \mathbb{N}$, with $B_{\ell,j} \in \Gamma_n$ and Borel partitions $\{\mathcal{Q}_{\ell,j}\}_{j=1}^{k_\ell}$ of Γ_n such that, as $\ell \rightarrow \infty$,

$$\sum_{j=1}^{k_\ell} B_{\ell,j}^t \mu(\mathcal{Q}_{\ell,j}) \longrightarrow \int_{\Gamma_n} A^t \mu(A), \quad \sum_{j=1}^{k_\ell} B_{\ell,j}^{t'} \mu(\mathcal{Q}_{\ell,j}) \longrightarrow \int_{\Gamma_n} A^{t'} \mu(A).$$

Due to the operator concavity of $x^{t'/t}$ on $(0, \infty)$, we have

$$\sum_{j=1}^{k_\ell} \mu(\mathcal{Q}_{\ell,j}) A^{t'} + \mu(\mathbb{P}_m \setminus \Gamma_n) I \leq \left(\sum_{j=1}^{k_\ell} \mu(\mathcal{Q}_{\ell,j}) A^t + \mu(\mathbb{P}_m \setminus \Gamma_n) I \right)^{\frac{t'}{t}}.$$

Letting $l \rightarrow \infty$ gives

$$\int_{\Gamma_n} A^{t'} d\mu(A) + \mu(\mathbb{P}_m \setminus \Gamma_n) I \leq \left(\int_{\Gamma_n} A^t d\mu(A) + \mu(\mathbb{P}_m \setminus \Gamma_n) I \right)^{\frac{t'}{t}}.$$

Since $\|A^t\|$ and $\|A^{t'}\|$ are integrable with respect to μ , (4.14) follows by taking the limit of the above inequality as $n \rightarrow \infty$. Then, (4.15) also follows by replacing μ with μ^{-1} in (4.14). Now, similarly to the proof of [5, Theorem 1] we see that for $1 \leq j \leq m$, as $r \geq t \searrow 0$, the j th eigenvalue of $(\int_{\mathbb{P}_m} A^t d\mu(A))^{\frac{1}{t}}$ is decreasing and that of $(\int_{\mathbb{P}_m} A^{-t} d\mu(A))^{-\frac{1}{t}}$ is increasing.

Furthermore, by applying Lemma 4.7 to μ^r we have

$$\left(\int_{\mathbb{P}_m} A^t d\mu^r(A) \right)^{\frac{1}{t}} \longrightarrow \exp \int_{\mathbb{P}_m} \log A d\mu^r(A) \quad \text{as } t \rightarrow 0 \text{ with } |t| \leq 1,$$

which is rephrased as

$$\left(\int_{\mathbb{P}_m} A^t d\mu(A) \right)^{\frac{1}{t}} \longrightarrow \exp \int_{\mathbb{P}_m} \log A d\mu(A) \quad \text{as } t \rightarrow 0 \text{ with } |t| \leq r.$$

Hence, as $r \geq t \searrow 0$, $|||(\int_{\mathbb{P}_m} A^t d\mu(A))^{\frac{1}{t}}|||$ decreases to $|||\exp \int_{\mathbb{P}_m} \log A d\mu(A)|||$ while $|||(\int_{\mathbb{P}_m} A^{-t} d\mu(A))^{-\frac{1}{t}}|||$ increases to the same. In view of (a) it remains to show the first inequality in (4.13). But this is immediately seen by applying (4.8) to μ^t for $0 < t \leq r$. \square

Remark 4.10. The following example shows that condition (4.9) is not satisfied for any $r > 0$ even if we have $\mu \in \mathcal{P}^p(\mathbb{P}_m)$ for all $p > 0$. For instance, choose $A_n \in \mathbb{P}_m$ such that $A_n \geq I$ and $\|A_n\| = n^n$, and define

$$\mu := \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_{A_n}.$$

Then, for any $r > 0$,

$$\int_{\mathbb{P}_m} \|A\|^r d\mu(A) = \sum_{n=1}^{\infty} \frac{(n^r)^n}{2^n} = \infty,$$

while

$$\int_{\mathbb{P}_m} \|\log A\|_2^p d\mu(A) \leq \sum_{n=1}^{\infty} \frac{(m \log^2 \|A_n\|)^{p/2}}{2^n} = m^{p/2} \sum_{n=1}^{\infty} \frac{(n \log n)^p}{2^n} < \infty$$

for all $p > 0$.

Problem 4.11. Do Theorem 4.8 and part (a) of Corollary 4.9 hold for general $\mu \in \mathcal{P}^2(\mathbb{P}_m)$ without assumption (4.9)? In part (b) of Corollary 4.9, we cannot define $\int_{\mathbb{P}_m} A^{\pm t} d\mu(A)$ for general $\mu \in \mathcal{P}^2$, while part (a) makes sense for general $\mu \in \mathcal{P}^2$.

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